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THE HASSE INVARIANT OF A VECTOR BUNDLE

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Abstract. The object of this work is to define, by analogy with algebra, the Witt group and the graded Brauer group of a topological space X. A homomorphism is defined between them analogous to the generalized Hasse invariant. Upon evaluation, the Witt group is seen to be $\tilde{K}O(X)$, the graded Brauer group $1+H^1(X; Z_2)+H^2(X; Z_2)$ with truncated cup product multiplication, while the homomorphism is given by Stiefel-Whitney classes: $1+w_1+w_2$.

Introduction. The Hasse invariant of a quadratic form on a vector space over a field k is an element of the Brauer group of k. It can be generalized to a homomorphism from the Witt group of classes of quadratic forms to the graded Brauer group of classes of graded k-algebras [3], [16]. Curiously enough, in this guise it is quite analogous to the homomorphism given by the first two Stiefel-Whitney classes which maps real vector bundles over a space X to the group $1+H^1(X; Z_2)+H^2(X; Z_2)$, where multiplication is by truncated cup product. The purpose of this paper is to display this analogy.

Delzant has defined Stiefel-Whitney classes of quadratic forms [4], [13] using the discriminant and Hasse invariant. It should add insight into his definition to see that the first two topological Stiefel-Whitney classes can also be defined in an algebraic way.

A vector bundle E is known to be orientable for real K-theory if its first two Stiefel-Whitney classes vanish. In this case it can be read off from the diagram of Theorem 3.1 that the Clifford bundle of E is a bundle of graded endomorphism algebras of a graded module bundle, which is then used to construct the Thom class of E. Donovan and Karoubi in studying this orientability question have done independently much of this same work, applying it further to define K-theory with local coefficients [6], [7].

The connection between algebra and topology comes from considering bundles over X whose fibers are vector spaces with quadratic forms or graded algebras and subjecting them to the same analysis the fiber alone is accorded in algebra. This is described in some detail in §I. §II is devoted to the algebraic computations and §III to a study of the topological problems.

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I. Let C be a category whose objects are real vector spaces, perhaps with additional structure, and whose morphisms are isomorphisms. Let $\perp$ be a binary operation defined on C which is associative and commutative. Then $(C, \perp)$ is called a *category with product*. A functor between two such is required to preserve the product. A subset of the objects of C is called *cofinal* if given any object P of C there is another one Q such that $P \perp Q$ is in the subset. The objects of a cofinal subset will be called *basic*.

We will now give the six examples of categories with product with which we shall be concerned.

1. The category of real vector spaces R^n with $n \geqq 0$ will be denoted by P. The product is direct sum $\oplus$, and we will consider all objects to be basic.

2. The category of real vector spaces R^n with $n > 0$ and product the tensor product $\otimes$ will be denoted FP. We will again take all objects to be basic.

3. Let q be a nondegenerate quadratic form defined on R^n with $n \geqq 0$. We let Q be the category of pairs (R^n, q), with isometries as morphisms and product the orthogonal direct sum. The object (R^n, q) is basic if q has signature zero.

4. We let Az denote the category of *Azumaya*, or *central simple*, algebras over the reals. The objects are then n-by-n matrix algebras over the reals or the quaternions: $M_n(R)$ or $M_n(H)$. The morphisms are algebra isomorphisms, the operation is the tensor product, and the $M_n(R)$ are basic.

5. We let FP_2 denote the category of Z_2-graded vector spaces $R^p \oplus R^q$ with $p, q \geqq 0$, $p+q > 0$. An element x of R^p is called *homogeneous of degree* 0 $(dx=0)$, while y of R^q is homogeneous with $dy=1$. An isomorphism f of $R^p \oplus R^q$ is required to be homogeneous of degree zero in the sense that it must send R^p to R^p and R^q to R^q. The product is the graded tensor product $\hat{\otimes}$; for objects, $(R^p \oplus R^q) \hat{\otimes} (R^s \oplus R^t) = R^{ps+qt} \oplus R^{qs+pt}$ while for morphisms, $(f \hat{\otimes} g)(x \hat{\otimes} y) = f(x) \hat{\otimes} g(y)$. An object $R^p \oplus R^q$ is basic if $p=q$.

6. We let Az_2 denote the category of Z_2-graded Azumaya algebras. These algebras A are expressible as $A = A^0 \oplus A^1$ with A^0 nonempty. An element of A^0 or A^1 is called *homogeneous* and we will let hA denote the set of homogeneous elements of A. If $a \in hA$, $da = i$ for $a \in A^i$. We have $d(ab) = da + db \pmod 2$ for $a, b \in hA$. Morphisms are required to preserve degree. The product is $\hat{\otimes}$, where multiplication in $A \hat{\otimes} B$ is given by $(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{db_1 da_2} a_1 a_2 \hat{\otimes} b_1 b_2$ for $a_i \in hA$, $b_i \in hB$. The product of morphisms is $(f \hat{\otimes} g)(a \hat{\otimes} b) = f(a) \hat{\otimes} g(b)$. Wall [16] has classified these algebras and they are listed in the Table. The basic objects are the graded endomorphism algebras (denoted END $(R^n \oplus R^n)$ or $M_{n,n}$ and defined later). This category includes the real Clifford algebras.

Now let X be a finite CW complex. We shall assume that X is connected, as this simplifies the discussion and generalizations to nonconnected spaces follow easily. If C is a category with product we define another category $C(X)$ whose objects are isomorphism classes of bundles over X. The fiber P of such a bundle is an object of C and the structural group is the automorphism group of P in C. Such bundles are in one-to-one correspondence with $[X, B\operatorname{Aut}(P)]$, where $B\operatorname{Aut}(P)$ is the classifying space of the automorphism group of P [11]. The product in C defines a product in $C(X)$, making $C(X)$ into a semigroup, which we then make into a group $KC(X)$ by the Grothendieck method. In $KC(X)$ the class of a bundle E will be denoted $[E]_C$, or just $[E]$ if there will be no confusion. If X is a point this is just the group K_0C as defined by Bass [3].

If $f\colon X \to Y$ is a continuous map, f induces the usual $f^*\colon KC(Y) \to KC(X)$. The inclusion of the basepoint into X induces a map $\operatorname{rk}\colon KC(X) \to K_0C$ whose kernel will be denoted $\tilde{K}C(X)$. Thus we get a split short exact sequence

$$0 \longrightarrow \tilde{K}C(X) \longrightarrow KC(X) \xrightarrow{\operatorname{rk}} K_0C \longrightarrow 0.$$

We will now prove that $\tilde{K}C(X)$ is isomorphic to the group of stable equivalence classes of bundles in $C(X)$.

Call two bundles E and F in $C(X)$ *stably equivalent* if there exist trivial bundles P_1 and P_2 such that $E \perp P_1$ and $F \perp P_2$ are equivalent. Let $\{E\}_C$ (or just $\{E\}$) be the stable equivalence class of E, and $TC(X)$ the set of classes. Note that we can always find a representative of $\{E\}$ whose fiber is basic. For if the fiber of E is P and $P \perp Q$ is basic, then the product of E and the trivial bundle of fiber Q has a basic fiber and lies in $\{E\}$. We will call a bundle *basic* if its fiber is basic.

PROPOSITION 1.1. *The set $TC(X)$ is a group, which is isomorphic to $\tilde{K}C(X)$.*

Proof. In showing that $TC(X)$ is a group, the only difficulty is the existence of inverses. The proof given by Milnor in [12, Theorem 4.1] for microbundles and adapted by Spivak in [14] for spherical fiber spaces will also work here for our operation $\perp$. The isomorphism from $TC(X)$ to $\tilde{K}C(X)$ is then given by

$$\{E\} \to [E] \perp [\operatorname{rk} E]^{-1}.$$

There are product preserving functors defined between these various categories which will unite their study into a single diagram. The *hyperbolic functor* $H\colon P \to Q$ is defined by $H(R^n) = (R^n \oplus R^{n*}, q)$ with $q(x, y) = y(x)$. If f is an isomorphism of R^n then $H(f) = f \oplus f^{*-1}$.

Next is End: $FP \to Az$ which assigns to R^n its endomorphism algebra $M_n(R)$. An automorphism γ of R^n goes to inner automorphism by γ on $M_n(R)$. The analogous graded functor END: $FP_2 \to Az_2$ assigns to $R^p \oplus R^q$ its graded endomorphism algebra. The homogeneous elements of

$$\mathrm{END}^0\,(R^p \oplus R^q) \quad \text{or} \quad \mathrm{END}^1\,(R^p \oplus R^q)$$

preserve or reverse degree, respectively. For simplicity of notation we use $M_{p,q}$ for END $(R^p \oplus R^q)$. A typical element is a matrix of $M_{p+q}(R)$ with grading indicated by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

If $f \in M_{p,q}$ and $g \in hM_{s,t}$ then

$$(f \hat{\otimes} g)(x \hat{\otimes} y) = (-1)^{dg\, dx} f(x) \hat{\otimes} g(y) \quad \text{for } x \in h(R^p \oplus R^q), \quad y \in R^s \oplus R^t.$$

This agrees with the previous definition if f and g are morphisms in the category FP_2; i.e. isomorphisms in END^0. An automorphism of $R^p \oplus R^q$ is sent by END to inner automorphism of $M_{p,q}$.

The functor $\Lambda: P \to FP_2$ sends R^n to $\Lambda^{\text{even}}R^n \oplus \Lambda^{\text{odd}}R^n$, where $\Lambda^{\text{even}}R^n$ is the sum of the even exterior powers of R^n and $\Lambda^{\text{odd}}R^n$ the odd. A linear transformation of R^n extends in the usual way to a morphism of ΛR^n.

And $\mathrm{Cl}: Q \to Az_2$ assigns to (R^n, q) the Clifford algebra it generates [2]. An isometry of (R^n, q) extends to an algebra automorphism of $\mathrm{Cl}\,(R^n, q)$.

THEOREM 1.2. (*See Bass* [3].) *The following diagram is commutative*:

$$\begin{array}{ccc} P & \xrightarrow{H} & Q \\ \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \mathrm{Cl}} \\ FP_2 & \xrightarrow{\mathrm{END}} & Az_2. \end{array}$$

Because these functors preserve products we get immediately a diagram

$$\tag{1} \begin{array}{ccccccc} K_0P & \xrightarrow{H} & K_0Q & \xrightarrow{\varphi} & Witt & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \mathrm{Cl}} & & \downarrow{\scriptstyle L} & & \\ K_0FP_2 & \xrightarrow{\mathrm{END}} & K_0Az_2 & \xrightarrow{\psi} & GB & \longrightarrow & 0, \end{array}$$

where *Witt* is the *Witt group* of the reals, and *GB* is the *graded Brauer group* as defined by Wall [16]. The diagram can be regarded as defining *Witt* and *GB*, with φ and ψ the natural projections onto the two cokernels. The induced map L is a generalization of the classical Hasse invariant.

Because these functors naturally extend to bundles we can define two more diagrams.

$$\tag{2} \begin{array}{ccccccc} KP(X) & \xrightarrow{H} & KQ(X) & \xrightarrow{\varphi} & Witt(X) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \mathrm{Cl}} & & \downarrow{\scriptstyle L} & & \\ KFP_2(X) & \xrightarrow{\mathrm{END}} & KAz_2(X) & \xrightarrow{\psi} & GB(X) & \longrightarrow & 0, \end{array}$$

and

$$
(3)\qquad
\begin{array}{ccccccc}
\tilde{K}P(X) & \xrightarrow{H} & \tilde{K}Q(X) & \xrightarrow{\varphi} & Witt^{\sim}(X) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \mathrm{Cl}} & & \downarrow{\scriptstyle L} & & \\
\tilde{K}FP_2(X) & \xrightarrow{\mathrm{END}} & \tilde{K}Az_2(X) & \xrightarrow{\psi} & GB^{\sim}(X) & \longrightarrow & 0.
\end{array}
$$

In (2), $Witt(X)$ and $GB(X)$ are defined to be the cokernels of H and END, and similarly in (3). We call $Witt^{\sim}(X)$ the *reduced Witt group* of X and $GB^{\sim}(X)$ the *reduced graded Brauer group*. Clearly diagram (2) is known if diagrams (1) and (3) are. It is the comparison of diagrams (1) and (3) which leads to the analogies mentioned in the introduction, and to our definition of L in (3) as a Hasse invariant.

II. This section is devoted to the calculation of the groups and maps of diagram (1).

THEOREM 2.1. *The values of the groups and maps in diagram* (1) *are*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \xrightarrow{H} & Z\oplus Z & \xrightarrow{\varphi} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \mathrm{Cl}} & & \downarrow{\scriptstyle L} & & \\
0 & \longrightarrow & Q_+ & \xrightarrow{\mathrm{END}} & K_0Az_2 & \xrightarrow{\psi} & Z_8 & \longrightarrow & 0
\end{array}
$$

where

(i) *The map H is the diagonal and $\varphi(s, t)=s-t$.*

(ii) *The lower row is a nonsplit extension with K_0Az_2 the index two subgroup of $Q_+\sqrt{2}\oplus Z_8$ ($Q_+\sqrt{2}$ is the positive rationals with $\sqrt{2}$ adjoined) given by all $(a/b, 2i)$ and $((a/b)\sqrt{2}, 2i+1)$. The group operation is multiplication in the first term and addition in the second.*

(iii) END $(a/b)=(a/b, 0)$ *and ψ is the projection onto the second term.*

(iv) $\Lambda(n)=2^n$, $\mathrm{Cl}\,(s, t)=(2^{(s+t)/2}, (s-t) \bmod 8)$, *and L is reduction* mod 8.

The proof follows in a sequence of propositions.

We establish the first row as follows. Since the objects of P correspond to the additive semigroup of nonnegative integers, we see that $K_0P\cong Z$.

A real nondegenerate quadratic form is given up to equivalence by a diagonal matrix of s (-1)'s and t $(+1)$'s, thus is completely described by the pair (s, t). Upon including additive inverses we get $K_0Q\cong Z\oplus Z$. As H is seen to be the diagonal map, *Witt* is isomorphic to Z, the map φ being $\varphi(s, t)=s-t$, the negative of the signature. (A choice made here for later convenience.)

Next we will compute End: $K_0FP\to K_0Az$ as an aid for the graded case. The classical *Brauer group* $B(R)$ is the cokernel of End.

PROPOSITION 2.2. *The evaluation of the sequence*

$$
K_0FP \xrightarrow{\mathrm{End}} K_0Az \longrightarrow B(R) \longrightarrow 0
$$

is $0 \to Q_+ \to Q_+ \times Z_2 \to Z_2 \to 0$ *which is split by the map* $Z_2 \to Q_+ \times Z_2$ *given by* $0 \to (1, 0)$, $1 \to (1, 1)$.

Proof. The objects of FP comprise the multiplicative semigroup of positive integers, so that $K_0FP \cong Q_+$, the positive rationals.

The objects of Az can be described as pairs: let $(n, 0)$ correspond to $M_n(R)$ and $(2n, 1)$ to $M_n(H)$. The first entry is the square root of the real dimension. If we handle the second entries additively and the first multiplicatively, Az is seen to be the semigroup $Z_+ \times Z_2$ and $K_0Az \cong Q_+ \times Z_2$, and $B(R) \cong Z_2$, with two elements, R and H.

We now begin the graded END sequence.

PROPOSITION 2.3. *Let* $\alpha: FP \to FP_2$ *send* R^n *to* $R^n \oplus R^0$. *Then* α *induces an isomorphism of* K_0FP *and* K_0FP_2.

Proof. Define the *forgetful functor* $\beta: FP_2 \to FP$ that sends $R^p \oplus R^q$ to R^{p+q}. Since $R^p \oplus R^q$ and $R^s \oplus R^t$ represent the same element of K_0FP_2 if and only if $p+q=s+t$ (tensor each with $R^1 \oplus R^1$) we see $R^{p+q} \oplus R^0$ and $R^p \oplus R^q$ are equivalent in K_0FP_2, thus α and β are inverse.

The algebras in Az_2 have been classified by Wall [16] who assigned to each a symbol, given in the Table, column 1. The corresponding algebra, ignoring the grading, is given in column 2. After placing the algebras in equivalence classes in GB, there are eight shortened symbols given in column 3. These, Wall showed, correspond to mod 8 integers which we indicate in column 4 (actually the inverse to Wall's correspondence), so that GB is isomorphic to Z_8. The entries in the symbols can best be explained in reference to an exact sequence and the diagram relating the End and END sequences.

The sequence goes as follows. Let $Q(R)$ be the set of 2-dimensional real algebras; i.e. $R \oplus R$ and C. Similarly $Q_2(R)$ is the set of graded 2-dimensional algebras and there are four of these as each of $R \oplus R$ and C can be graded in two different ways. We can include $Q(R)$ in $Q_2(R)$ by concentrating in degree zero, and then using the group operations given by Bass, we have the

PROPOSITION 2.4 [3, CHAPTER 4, PROPOSITION 3.3]. *The sequence*

$$0 \to Q(R) \to Q_2(R) \to Z_2 \to 0$$

is exact, and is evaluated

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0.$$

For the diagram, we first define $\alpha': Az \to Az_2$ by sending $M_n(R)$ to $M_{n,0}(R)$ and $M_n(H)$ to $M_{n,0}(H)$; that is, to themselves but graded in degree zero. The maps α and α' are product preserving, and $\mathrm{END} \circ \alpha = \alpha' \circ \mathrm{End}$.

Proposition 2.5. *The following diagram is exact and commutative*:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0FP & \xrightarrow{\text{End}} & K_0Az & \longrightarrow & B(R) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow & & \\
0 & \longrightarrow & K_0FP_2 & \xrightarrow{\text{END}} & K_0Az_2 & \longrightarrow & GB & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & Q_2(R) & \longrightarrow & Q_2(R) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Proof. The right-hand column is proved by Bass [3, Chapter 4, Theorem 4.4]. The maps α' and END are monomorphisms, as will be evident after an explanation of the symbols. That coker $\alpha' \cong Q_2(R)$ follows from the rest of the diagram.

We can now explain the symbols. The R or H refers to an element of $B(R)$, the $+$ or $-$ to the right Z_2 in Proposition 2.4, the $+1$ or -1 to $Q(R)$. Then n refers to the size of the matrices as in the Table, and the (p, q) in cases 0 and 4 to the grading as END $(R^p \oplus R^q)$ or END $(H^p \oplus H^q)$. This last bit of information is lost in K_0Az_2 for the same reason it was in K_0FP_2 (Proposition 2.3), so we can ignore it. Building from the symbols to K_0Az_2 is then seen to involve three group extensions, two of which are already done.

Proof of Theorem 2.1. Column 5 of the Table gives a pair which consists of the square root of the real dimension of the algebra and its class in GB, which makes

Wall's symbols	*Ungraded algebras*	*Shortened symbols*	Z_8	
$(+, R, +1, n, (p, q))$	$M_n(R)$	$(+, R, +1)$	0	$(n, 0)$
$(-, R, +1, n)$	$M_n(R) \oplus M_n(R)$	$(-, R, +1)$	7	$(n\sqrt{2}, 7)$
$(+, R, -1, 2n)$	$M_{2n}(R)$	$(+, R, -1)$	6	$(2n, 6)$
$(-, H, -1, n)$	$M_{2n}(C)$	$(-, H, -1)$	5	$(2n\sqrt{2}, 5)$
$(+, H, +1, n, (p, q))$	$M_n(H)$	$(+, H, +1)$	4	$(2n, 4)$
$(-, H, +1, n)$	$M_n(H) \oplus M_n(H)$	$(-, H, +1)$	3	$(2n\sqrt{2}, 3)$
$(+, H, -1, n)$	$M_n(H)$	$(+, H, -1)$	2	$(2n, 2)$
$(-, R, -1, n)$	$M_n(C)$	$(-, R, -1)$	1	$(n\sqrt{2}, 1)$

Table

clear the semigroup structure in Az_2 as multiplication in the first factor and mod 8 addition in the second. When we include inverses, we get the group stated in the theorem. The maps are deduced easily from the definitions, except perhaps Cl. The dimension of $\mathrm{Cl}(s, t)$ is 2^{s+t} while its graded Brauer class comes from an inspection of the algebras as described in [2]. One can note also that in this notation $\alpha'(a/b, i)=(a/b, 4i)$.

III. In this section we investigate the reduced cohomology theories $\tilde{K}C$ to find their classifying spaces and evaluate diagram (3).

THEOREM 3.1. *The values of the groups and maps in diagram* (3) *are*

$$\begin{array}{ccccccc} 0 \to \tilde{K}O(X) & \xrightarrow{H} & \tilde{K}O(X) \oplus \tilde{K}O(X) & \xrightarrow{\varphi} & \tilde{K}O(X) \to 0 & & \\ \downarrow \Lambda & & \downarrow \mathrm{Cl} & & \downarrow L & & \\ 0 \to \tilde{K}FP_2(X) & \xrightarrow{\mathrm{END}} & \tilde{K}Az_2(X) & \xrightarrow{\psi} & 1 \oplus H^1(X; Z_2) \oplus H^2(X; Z_2) \to 0, & & \end{array}$$

where:

(i) *The map H is the diagonal and $\varphi(\{E\}, \{F\})=\{E\}-\{F\}$.*

(ii) *There is an isomorphism between $\tilde{K}FP_2(X)$ and $Q \otimes \tilde{K}O(X)$.*

(iii) $\psi \circ \mathrm{Cl}(\{E\}, \{F\})=(1+w_1(E)+w_2(E))(1+w_1(F)+w_1(F)^2+w_2(F))$, *so that* $L(\{E\})=1+w_1(E)+w_2(E)$.

Again we begin with the first row. Clearly $\tilde{K}P(X)$ is $\tilde{K}O(X)$.

A basic bundle in $Q(X)$ has group the isometry group for some n of the quadratic form $-x_1^2-x_2^2-\cdots-x_n^2+y_1^2+\cdots+y_n^2$. Since this group contains $O(n)\times O(n)$ as a deformation retract [9, p. 345], every bundle splits into a sum $E \oplus F$, where E has a negative definite and F a positive definite metric. So we can define $Witt^{\sim}(X)$ and φ as indicated.

As before we shall first study End: $\tilde{K}FP(X) \to \tilde{K}Az(X)$ as an aid for the graded case. We shall call its cokernel $B^{\sim}(X)$ the *reduced* (*ungraded*) *Brauer group* of X.

PROPOSITION 3.2. *The classifying space for FP is $BO_{\otimes}$, where $O_{\otimes}$ is the direct limit of the orthogonal groups $O(n)$ under the mappings $\bigotimes I_m: O(n) \to O(nm)$. There is a commutative diagram of isomorphisms*

$$\begin{array}{ccc} \tilde{K}FP(X) & \xrightarrow{ph'} & 1+\tilde{H}^{4*}(X; Q) \\ \downarrow & & \downarrow \ln \\ Q \otimes \tilde{K}O(X) & \xrightarrow{ph} & \tilde{H}^{4*}(X; Q). \end{array}$$

Proof. The first statement is clear, since if E is an n-plane bundle and ε the trivial line bundle, the group of $E \otimes m\varepsilon$ is the image of $O(n)$ in $O(nm)$. In matrix notation, if $e_1, \ldots, e_n$ is a basis for R^n and $d_1, \ldots, d_m$ a basis for R^m, and if we arrange the basis for $R^n \otimes R^m$ in reverse lexicographical order $e_1 \otimes d_1$, $e_2 \otimes d_1$,

$\ldots, e_n \otimes d_m$, the matrix of $A \otimes I_m$ for $A \in O(n)$ is given by A m times down the diagonal.

Define an isomorphism $\delta: \tilde{K}FP(X) \to 1 + Q \otimes \tilde{K}O(X)$ [3, p. 40] by $\delta\{E\}_{FP} = 1 + 1/\mathrm{rk}\, E \otimes \{E\}_P$. This map is well defined and preserves multiplication. It is surjective, since given any element of $Q \otimes \tilde{K}O(X)$, it can first be written in the form $1/n \otimes \{E\}_P$ with $n > 0$. Then choose r such that $rn > \dim X$ and rewrite it as

$$1/rn \otimes \{E \otimes r\varepsilon\}_P = 1/\mathrm{rk}\, F \otimes \{F\}_P,$$

for that unique bundle F of rank rn in this stable equivalence class. Then $\delta\{F\}_{FP} = 1 + 1/n \otimes \{E\}_P$. And δ is injective, since if $1 + 1/\mathrm{rk}\, E \otimes \{E\}_P = 1 + 1/\mathrm{rk}\, F \otimes \{F\}_P$, taking E and F of the same dimension $n > \dim X$ we have $1/n \otimes (\{E\}_P - \{F\}_P) = 0$, hence $m\{E\}_P = m\{F\}_P$ for some m. Then $E \otimes m\varepsilon = F \otimes m\varepsilon$ or $\{E\}_{FP} = \{F\}_{FP}$.

This map δ followed by ln (given by the series expansion of $\ln(1+x)$) gives the isomorphism on the left, while ln itself is used on the right.

The *Pontrjagin character* $\mathrm{ph}: Q \otimes \tilde{K}O(X) \to \sum_{i \geq 1} H^{4i}(X; Q)$ is an isomorphism [10] and because it is multiplicative we can define $\mathrm{ph}' \{E\}_{FP} = \mathrm{ph}\, E/\mathrm{rk}\, E$. The diagram then commutes, for

$$\begin{aligned} \mathrm{ph} \ln \delta\{E\}_{FP} &= \ln\left(1 + \mathrm{ph}\left(\frac{1}{\mathrm{rk}\, E} \otimes \{E\}_P\right)\right) = \ln\left(1 + \frac{\mathrm{ph}\, \{E\}}{\mathrm{rk}\, E}\right) \\ &= \ln\left(1 + \frac{\mathrm{ph}\, E - \mathrm{rk}\, E}{\mathrm{rk}\, E}\right) = \ln \mathrm{ph}' \{E\}_{FP}. \end{aligned}$$

This completes the proof.

A basic bundle in $Az(X)$ has fiber $M_n(R)$. Since by the Skolem-Noether theorem all automorphisms of $M_n(R)$ are inner, the structural group of the bundle is the *projective orthogonal group* $PO(n)$, which is the quotient of $O(n)$ by its center Z_2. Furthermore if E is a vector bundle with fiber R^n then End E has fiber $M_n(R)$ and the same coordinate transformations acting by inner automorphism. Thus if E is classified by $f: X \to BO(n)$, we see that End E is classified by f followed by the natural projection into $BPO(n)$.

PROPOSITION 3.3. *The classifying space for Az is BPO, where PO is the limit of the projective orthogonal groups under* $\otimes I_m: PO(n) \to PO(nm)$. *The projection* $O(n) \to PO(n)$ *induces* End: $\tilde{K}FP(X) \to \tilde{K}Az(X)$ *whose cokernel* $B^{\sim}(X) \cong H^2(X; Z_2)$. *Thus the following sequence is exact*:

$$0 \longrightarrow \tilde{K}FP(X) \xrightarrow{\mathrm{End}} \tilde{K}Az(X) \longrightarrow H^2(X; Z_2) \longrightarrow 0.$$

Proof. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_2 & \longrightarrow & O(n) & \longrightarrow & PO(n) & \longrightarrow & 0 \\ & & \| & & \downarrow \otimes I_m & & \downarrow \otimes I_m & & \\ 0 & \longrightarrow & Z_2 & \longrightarrow & O(nm) & \longrightarrow & PO(nm) & \longrightarrow & 0 \end{array}$$

is exact and commutative, hence in the limit there is a short exact sequence $0 \to Z_2 \to O_\otimes \to PO \to 0$. This gives a sequence of fiber spaces $Z_2 \to O_\otimes \to PO \to K(Z_2, 1) \to BO_\otimes \to BPO \to K(Z_2, 2)$. As $Z_2 \to O(n)$ for n even is homotopic to the constant map, it follows that $K(Z_2, 1) \to BO_\otimes$ is likewise, so that End is a monomorphism. The Bott Periodicity Theorem and our direct limit system give the homotopy of $BO_\otimes$ as Q in every dimension $4i$, $i > 0$. Hence there can be no obstruction to a cross-section of $BPO \to K(Z_2, 2)$ and we see that

$$\tilde{K}Az(X) \to H^2(X; Z_2)$$

is onto.

REMARK. For the complex numbers, $B^{\sim}(X)$ is the torsion subgroup of $H^3(X; Z)$. See [8].

Added in proof. A proof that $\tilde{K}Az(X) \to H^2(X; Z_2)$ is onto which avoids the use of the Bott Periodicity Theorem is given in [6]. They show that End is naturally split by the map $\frac{1}{2}j\colon \tilde{K}Az(X) \to \tilde{K}FP(X)$, where j is induced by the map of $PO(n)$ to $O(n^2)$ sending $\pm\alpha$ to $\alpha \otimes \alpha$. Lemma 3.11 shows that for spheres, $\tilde{K}Az$ and $\tilde{K}FP \oplus H^2$ agree, and hence by 7.1 of [5], they agree for all finite complexes.

We now begin the graded END sequence.

PROPOSITION 3.4. $\tilde{K}FP_2(X) \cong \tilde{K}FP(X)$.

Proof. As with X a point (Proposition 2.3), take $\alpha\colon FP(X) \to FP_2(X)$ given by $\alpha(E) = E \oplus 0$ and $\beta\colon FP_2(X) \to FP(X)$ the forgetful functor. The proof is the same.

Before we can properly discuss stabilizing the bundles in $Az_2(X)$ we need a matrix formula for the graded tensor product of $M_{n,m}$ and $M_{p,q}$. If the basis for $R^n \oplus R^m$ is $e_1, \ldots, e_n, e'_1, \ldots, e'_m$, and for $R^p \oplus R^q$ is $d_1, \ldots, d_p, d'_1, \ldots, d'_q$, we must choose the basis order for $R^{np+mq} \oplus R^{mp+nq}$. It comes naturally in four subsets, each with the reverse lexicographical ordering:

$$\begin{gathered} e_1 \otimes d_1, e_2 \otimes d_1, \ldots, e_n \otimes d_p \\ e'_1 \otimes d'_1, \ldots, e'_m \otimes d'_q \\ e'_1 \otimes d_1, \ldots, e'_m \otimes d_p \\ e_1 \otimes d'_1, \ldots, e_n \otimes d'_q. \end{gathered}$$

LEMMA 3.5. *With this choice of basis order we have the following formulas for the graded tensor product of two elements of $M_{1,1}$ and for an element of $M_{n,m}$ and the identity matrix $I_{p,q}$.*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \left(\begin{array}{cc|cc} ae & -bf & be & af \\ cg & dh & -dg & ch \\ \hline ce & -df & de & cf \\ ag & bh & -bg & ah \end{array}\right),$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \hat{\otimes} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \left(\begin{array}{cc|cc} A \otimes I_p & 0 & B \otimes I_p & 0 \\ 0 & D \otimes I_q & 0 & C \otimes I_q \\ \hline C \otimes I_p & 0 & D \otimes I_p & 0 \\ 0 & B \otimes I_q & 0 & A \otimes I_q \end{array}\right).$$

Proof. Evaluate these linear transformations on the basis elements.

REMARK. Unfortunately, due to the choice of ordering of the basis elements, this product is not associative. It would be associative if we did not separate the basis elements into subsets, but this would make the grading confusing.

Define the *graded orthogonal group* $GO(n)$ to be all matrices in $O(2n)$ of the form

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$$

with $A, B \in O(n)$. Define the *projective graded orthogonal group* $PGO(n)$ to be $GO(n)$ modulo $\pm I_{n,n}$.

PROPOSITION 3.6. *The classifying space for Az_2 is $BPGO$, the limit of the projective graded orthogonal groups under $\hat{\otimes}\, I_{p,q}\colon PGO(n) \to PGO(n(p+q))$.*

Proof. A basic bundle in $Az_2(X)$ has fiber $M_{n,n}$. We need the structural group. Automorphisms of $M_{n,n}$ are all inner, so we must find the units which preserve the grading. In $M_{n,n}$ are two central idempotents,

$$u_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

which must be preserved or reversed by any inner automorphism. If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ sends u_1 to itself, then $B = C = 0$. If it sends u_1 to u_2, then $A = D = 0$. Hence the *homogeneous* units, modulo their center R^*, give all the inner automorphisms of $M_{n,n}$.

We wish to argue that stably we may take the structural group to be $PGO(2n)$. First for reasons of homotopy we can assume that we have only the units $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$ with $A, B, C, D \in O(n)$. We stabilize the bundle once by tensoring with END $(\varepsilon \oplus \varepsilon)$. The lemma then says our coordinate transformations $g_{ij}(x)$ have matrices of the form

$$\left(\begin{array}{cc|cc} A & & & \\ & D & & \\ \hline & & D & \\ & & & A \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|cc} & & B & \\ & & & C \\ \hline C & & & \\ & B & & \end{array}\right).$$

Let

$$J = \left(\begin{array}{c|cc} I_{2n} & & \\ \hline & & I_n \\ & I_n & \end{array}\right).$$

The bundle with coordinate transformations $g'_{ij}(x)=Jg_{ij}(x)J$ is equivalent to our bundle [15, 2.10] and here the $g'_{ij}(x)$ have matrices of the form

$$\left(\begin{array}{cc|cc} A & & & \\ & D & & \\ \hline & & A & \\ & & & D \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|cc} & & & B \\ & & C & \\ \hline & B & & \\ C & & & \end{array}\right)$$

and so are in $GO(2n)$.

For $GO(n)$ the matrix formulas for $\hat{\otimes}$ reduce to

$$\left(\begin{array}{c|c} A & \\ \hline & A \end{array}\right) \hat{\otimes}\, I_{p,q} = \left(\begin{array}{c|c} A \otimes I_{p+q} & \\ \hline & A \otimes I_{p+q} \end{array}\right)$$

and

$$\left(\begin{array}{c|c} & B \\ \hline B & \end{array}\right) \hat{\otimes} I_{p,q} = \left(\begin{array}{c|c} & B \otimes I_{p+q} \\ \hline B \otimes I_{p+q} & \end{array}\right).$$

Clearly $\hat{\otimes}\, I_{p,q}$ and $\hat{\otimes}\, I_{p',q'}$ give the same map if $p+q=p'+q'$ and if $\hat{\otimes}\, I_{p,q}$ is followed by $\hat{\otimes}\, I_{r,s}$, the result is the same as $\hat{\otimes}\, I_{pr+qs,qr+ps}$. This completes the proof.

The following commutative diagram of exact sequences of groups is necessary for the derivation of the END sequence. Let $\Delta\colon O(n) \to GO(n)$ send A to

$$\left(\begin{array}{c|c} A & \\ \hline & A \end{array}\right),$$

and let Δ' be the induced map, $\Delta'\colon PO(n) \to PGO(n)$.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_2 & \longrightarrow & O(n) & \longrightarrow & PO(n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta'} & & \\ 0 & \longrightarrow & Z_2 & \longrightarrow & GO(n) & \longrightarrow & PGO(n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & Z_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{4}$$

Since these maps commute with our direct limit system, we get in the limit the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_2 & \longrightarrow & O_{\otimes} & \longrightarrow & PO & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta'} & & \\
0 & \longrightarrow & Z_2 & \longrightarrow & GO_{\otimes} & \longrightarrow & PGO & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & Z_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{5}$$

The first row of (5) we have already studied in connection with End. The right-hand column is next. Recall the map $\alpha': Az \to Az_2$ which sends $M_n(R)$ to $M_{n,0}$. It defines a map of bundles, $\alpha': Az(X) \to Az_2(X)$.

PROPOSITION 3.7. *The map $\alpha': Az(X) \to Az_2(X)$ is induced by $\Delta': PO \to PGO$ and the following sequence is exact*:

$$0 \longrightarrow \tilde{K}Az(X) \xrightarrow{\alpha'} \tilde{K}Az_2(X) \xrightarrow{T_*} H^1(X; Z_2) \longrightarrow 0.$$

Proof. A bundle E in $Az(X)$ with fiber $M_n(R)$ is to be thought of as having fiber $M_{n,0}$. Then we form $E \mathbin{\hat{\otimes}} \text{END}\,(\varepsilon \oplus \varepsilon)$. If a coordinate transformation of E is $g_{ij}(x) = A \in PO(n)$, the new bundle has in its place

$$\left(\begin{array}{c|c} A & \\ \hline & A \end{array}\right) = \Delta'(A)$$

in $PGO(n)$.

The right-hand column of diagram (5) gives us the following sequence of fibrations:

$$Z_2 \longrightarrow BPO \xrightarrow{\Delta'_*} BPGO \xrightarrow{T} K(Z_2, 1),$$

which gives the exactness of the sequence of the proposition except at the ends. Because BPO is connected, $Z_2 \to BPO$ is homotopic to zero, thus α' is a monomorphism. The map of $GO(n)$ to Z_2 in (4) is split by sending 0 to

$$\left(\begin{array}{c|c} I & \\ \hline & I \end{array}\right)$$

and 1 to

$$\left(\begin{array}{c|c} & I \\ \hline I & \end{array}\right).$$

This induces a cross-section to T, implying that T_* is onto.

REMARK 3.8. If E is a graded algebra bundle, we can think of $T_*\{E\}$ as a line bundle with coordinate transformation $+1$ or $-1 \in O(2)$ where E has coordinate transformation

$$\left(\begin{array}{c|c} A & \\ \hline & A \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} & B \\ \hline B & \end{array}\right),$$

respectively.

PROPOSITION 3.9. *There is a one-to-one correspondence between $GB^{\smile}(X)$ and $H^1(X; Z_2) \times H^2(X; Z_2)$. Thus the following sequence is exact*: (*as sets at $H^1 \times H^2$*)

$$0 \longrightarrow \tilde{K}FP_2(X) \xrightarrow{\text{END}} \tilde{K}Az_2(X) \xrightarrow{\psi} H^1(X; Z_2) \times H^2(X; Z_2) \longrightarrow 0.$$

Proof. Diagram (5) leads to the following diagram, after checking surjectivity and injectivity of a few more maps:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \tilde{K}FP(X) & \xrightarrow{\text{End}} & \tilde{K}Az(X) & \longrightarrow & H^2(X; Z_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha'} & & \| & & \\
0 & \longrightarrow & [X, BGO_{\otimes}] & \longrightarrow & \tilde{K}Az_2(X) & \xrightarrow{U_*} & H^2(X; Z_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle T_*} & & & & \\
 & & H^1(X; Z_2) & = & H^1(X; Z_2) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We extract from this a diagonal sequence

$$\tilde{K}FP(X) \xrightarrow{\alpha' \circ \text{End}} \tilde{K}Az_2(X) \xrightarrow{T_* \times U_*} H^1(X; Z_2) \times H^2(X; Z_2),$$

which by definition and diagram chase yields the proposition.

There are two possible group structures for $H^1 \times H^2$. This is seen by computing $[X \wedge X, X]$ for $X = K(Z_2, 1) \times K(Z_2, 2)$ [1]. We shall compute the maps Cl and L of diagram (3) and in the process discover which group is present. The outline of this is as follows.

By taking Grassmann manifolds as test spaces and using naturality we see that the map L must be expressible in terms of Stiefel-Whitney classes. We will show by considering the Möbius bundle M over S^1 that w_1 is present, and by considering the Hopf bundle η over S^2 that w_2 is present. Then the map L must be either (w_1, w_2) or $(w_1, w_1^2+w_2)$. As neither of these is a homomorphism into $H^1(X; Z_2) \oplus H^2(X; Z_2)$ we conclude that the group present is $1 \oplus H^1(X; Z_2) \oplus H^2(X; Z_2)$. We then just have to consider these two possibilities for the map.

We begin by examining $\mathrm{Cl}\,(1, 1)$. Let d_1 and d_2 be a basis for $R^1 \oplus R^1$ with the quadratic form $q(ad_1+bd_2) = -a^2+b^2$. Then $\mathrm{Cl}\,(1, 1)$ has an algebra basis d_1, d_2 with $d_1^2 = -1$, $d_2^2 = +1$, and is isomorphic to $M_{1,1}$ if we use the correspondence

$$d_1 \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad d_2 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, Cl takes the four isometries of $O(1)\times O(1)$ of $R^1 \oplus R^1$ to inner automorphisms of $\mathrm{Cl}\,(1, 1)$ and of $M_{1,1}$ as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \pm 1 \to \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \to \pm d_2 \to \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \pm d_1 \to \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \to \pm d_1 d_2 \to \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

PROPOSITION 3.10. *The map* $L\colon \widetilde{KO}(S^1) \to H^1(S^1; Z_2)$ *sends the generator* $\{M\}$ *of* $\widetilde{KO}(S^1)$ *to* $w_1(M)$, *the nonzero element of* $H^1(S^1; Z_2)$.

Proof. We consider the bundle $M \oplus \varepsilon$ over S^1, with M having a negative definite and ε a positive definite metric. Let x and y be the two points of $S^0 \subset S^1$. Then the clutching function for $M \oplus \varepsilon$ sends x to $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ in $O(1)\times O(1)$ and y to $(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$. Therefore the clutching function for $\mathrm{Cl}\,(M \oplus \varepsilon)$ sends x to inner automorphism by $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ and y to inner automorphism by $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. This graded algebra bundle defines the line bundle M over S^1 again, as its clutching functions are $x \to +1$ and $y \to -1$ (cf. 3.8). By the usual correspondence between line bundles and $H^1(S^1; Z_2)$ we get $w_1(M)$.

Next we discuss $\mathrm{Cl}\,(2, 2)$. Suppose d_1, d_2, d_3, d_4 are the basis for $R^2 \oplus R^2$ with the quadratic form $q(ad_1+bd_2+cd_3+dd_4) = -a^2-b^2+c^2+d^2$. Then $\mathrm{Cl}\,(2, 2)$ has algebra basis elements d_1, d_2, d_3, d_4 with $d_1^2 = d_2^2 = -1$ and $d_3^2 = d_4^2 = +1$. To get matrix formulas we use the fact that $\mathrm{Cl}\,(2, 2) \cong \mathrm{Cl}\,(1, 1) \hat{\otimes} \mathrm{Cl}\,(1, 1)$, so that generators of $\mathrm{Cl}\,(2, 2)$ are given in terms of those of $\mathrm{Cl}\,(1, 1)$ by $d_1 = d_1 \hat{\otimes} 1$, $d_2 = 1 \hat{\otimes} d_1$, $d_3 = d_2 \hat{\otimes} 1$, $d_4 = 1 \hat{\otimes} d_2$. Using this, the matrix interpretation of $\mathrm{Cl}\,(1, 1)$, and Lemma 3.5, we get an isomorphism between $\mathrm{Cl}\,(2, 2)$ and $M_{2,2}$:

$$d_1 \to \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right), \quad d_2 \to \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right),$$

$$d_3 \to \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right), \quad d_4 \to \left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right).$$

LEMMA 3.11. *The fundamental group of PO is Z_2 and of $PO(n)$ is given by*

$$\begin{aligned} \pi_1 PO(n) &= Z, & & n = 2, \\ &= Z_2, & & n \text{ odd}, \quad n > 1, \\ &= Z_4, & & n \equiv 2 \pmod 4, \quad n > 2, \\ &= Z_2 \oplus Z_2, & & n \equiv 0 \pmod 4, \quad n > 2. \end{aligned}$$

Proof. The fibrations $Z_2 \to O_{\otimes} \to PO$ and $Z_2 \to O(n) \to PO(n)$ show all but, in the case n even and $n > 2$, which group of order 4 is present. In this case we can restrict to $SO(n)$ and $PSO(n)$. The simply connected covering space of $PSO(n)$ is Spin (n), and the fiber over the identity is the subgroup $\{1, -1, \omega, -\omega\}$ where ω is the product of all the algebra generators of Cl $(n, 0)$. As

$$\begin{aligned} \omega^2 &= +1, & & n \equiv 0 \pmod 4, \\ &= -1, & & n \equiv 2 \pmod 4) \end{aligned}$$

we arrive at the group claimed.

PROPOSITION 3.12. *The map $L: \widetilde{KO}(S^2) \to H^2(S^2; Z_2)$ sends the generator $\{\eta\}$ of $\widetilde{KO}(S^2)$ to $w_2(\eta)$, the nonzero element of $H^2(S^2; Z_2)$.*

Proof. We consider the bundle $\eta \oplus 2\varepsilon$ over S^2, with η having a negative definite and 2ε a positive definite metric. Since by Proposition 3.9 $\widetilde{KAz}_2(S^2) \cong GB^{\sim}(S^2) \cong Z_2$, and $\widetilde{KAz}_2(S^2) \cong \pi_1(PGO)$, it suffices to show that the clutching function of Cl $(\eta \oplus 2\varepsilon)$ represents the nonzero element of $\pi_1(PGO)$.

The clutching function $g: S^1 \to SO(2)$ for η is given by

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad 0 \leqq \theta < 2\pi.$$

Hence that of $\eta \oplus 2\varepsilon$ is

$$\left(\begin{array}{c|c} g(\theta) & \\ \hline & I_2 \end{array}\right)$$

in $O(2)\times O(2)$. By Cl these isometries of $R^2 \oplus R^2$ give rise to automorphisms of Cl (2, 2). By direct computation on basis elements one sees that Cl must take

$$\left(\begin{array}{c|c} g(\theta) & \\ \hline & I_2 \end{array}\right)$$

to inner automorphism of Cl (2, 2) by $\cos \theta/2+\sin (\theta/2)d_1d_2$, thus to inner automorphism of $M_{2,2}$ by

$$\left(\begin{array}{c|c} g(\theta/2) & \\ \hline & g(\theta/2) \end{array}\right).$$

Let $h(\theta)=g(\theta/2)$. Clearly the generator of $\pi_1 PSO(2)$ which is not hit by the projection from $\pi_1 SO(2)$ is $[h]$.

Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1 SO(2) & \longrightarrow & \pi_1 PSO(2) & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \otimes I_2} & & \downarrow{\scriptstyle \otimes I_2} & & \| & & \\
0 & \longrightarrow & \pi_1 SO(4) & \longrightarrow & \pi_1 PSO(4) & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \otimes I_n} & & \downarrow{\scriptstyle \otimes I_n} & & \| & & \\
0 & \longrightarrow & \pi_1 SO(4n) & \longrightarrow & \pi_1 PSO(4n) & \longrightarrow & Z_2 & \longrightarrow & 0.
\end{array}$$

The groups in the left column are all Z_2 except $\pi_1 SO(2)=Z$, while the center column is given by Lemma 3.11. In the left column, $\otimes I_n$ is zero for n even, an isomorphism for n odd. In the center column it follows that $[h]$ goes nontrivially to $\pi_1 PSO(4)$ and in fact goes to the generator of $\pi_1 PO$ in the limit. As $\Delta'(h(\theta))$ is the clutching function for Cl $(\eta \oplus 2\varepsilon)$, it follows since α' is a monomorphism that $\{\text{Cl}\,(\eta \oplus 2\varepsilon)\}$ is nonzero.

Proposition 3.13. *The maps* Cl *and* L *are given by*

$$\text{Cl}\,(\{E\},\{F\}) = (1+w_1(E)+w_2(E))(1+w_1(F)+w_1(F)^2+w_2(F))$$

and $L(\{E\})=1+w_1(E)+w_2(E)$.

Proof. We use the canonical line bundle ζ over the real projective plane. As in Proposition 3.10, the coordinate transformations for Cl $(\zeta \oplus \varepsilon)$ take values either $\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $\pm\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ in $PGO(1)$. If we construct a bundle using in corresponding places the coordinate transformations $+\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $+\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ we get a bundle in $[X, BGO(1)]$ which maps to Cl $(\zeta \oplus \varepsilon)$, showing that Cl $(\zeta \oplus \varepsilon)$ must map to zero in $H^2(RP^2; Z_2)$. We conclude that Cl $(\zeta \oplus \varepsilon)$ maps to $1+w_1(\zeta)$ in $GB^{\sim}(RP^2)$. As Cl $(\zeta \oplus \zeta)$ must map to 1 in $GB^{\sim}(RP^2)$, it follows that Cl $(\varepsilon \oplus \zeta)$ maps to $1+w_1(\zeta)+w_1(\zeta)^2$. This completes the proof of Theorem 3.1.

Parallel to Proposition 2.5 we have

COROLLARY 3.14. *The following diagram is exact and commutative.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}FP(X) & \xrightarrow{\text{End}} & \tilde{K}Az(X) & \longrightarrow & H^2(X;Z_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}FP_2(X) & \xrightarrow{\text{END}} & \tilde{K}Az_2(X) & \longrightarrow & 1+H^1(X;Z_2)+H^2(X;Z_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H^1(X;Z_2) & \longrightarrow & H^1(X;Z_2) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Finally, using $X = RP^2$ one sees that the vertical sequences need not split.

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